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Unimodular hermitian lattices in dimension 13

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Abstract

All indecomposable unimodular hermitian lattices in dimension 13 over the ring of integers in $\mathbb{Q}(\sqrt{-3})$ are determined.

Keywords: Hermitian lattice; Unimodular lattice

1. Introduction

In 1978 W. Feit [4] described unimodular hermitian lattices of dimensions up to 12 over the ring of integers in $\mathbb{Q}(\sqrt{-3})$. Our short note can be considered as a supplement to that work. We have determined the list of all unimodular lattices in dimension 13. There are exactly 14 (up to isomorphism) unimodular indecomposable lattices and they are described in Table 1. Unimodular lattices of rank at most 12 all have roots. Dimension 13 is the first case when a unimodular root-free lattice appears [1]. There is only one unimodular root-free lattice L_0 of rank 13. It has minimum norm 3 and its automorphism group is isomorphic to the group $\mathbb{Z}_6 \times \mathrm{PSp}_6(3)$ of order $2^{10} \cdot 3^{10} \cdot 5 \cdot 7 \cdot 13$. Remaining lattices have roots and all root systems are of rank 12.

Another unimodular lattice without roots was constructed in [2] by making use codes over rings. Our classification theorem shows that this lattice is isometric to L_0.

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Table 1
Unimodular lattices of rank 13

No	L	$L^{(2)}$	$\lvert G(L)\rvert$	$\mu_2(L)$
1	L_0	$\varnothing$	$2^{10}\cdot 3^{10}\cdot 5\cdot 7\cdot 13$	0
2	$\left\langle A_{12}\oplus\langle x\rangle,\ \frac{1}{13}(-12,1,\ldots,1)+\frac{3\sqrt{-3}}{13}x\right\rangle$, $(x,x)=13$	A_{12}	$2^{11}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\cdot 13$	468
3	$\left\langle A_8\oplus D_4(\sqrt{-3})\oplus\langle x\rangle,\ y_1+y_2+\frac{2+\sqrt{-3}}{9}x,\ \sqrt{-3}\,(y_2+\frac{1}{3}x)\right\rangle$, $y_1=\frac{1}{9}(-8,1,\ldots,1)$, $y_2=\frac{1}{\sqrt{-3}}(1,1,1,1)$, $(x,x)=9$	$A_8\oplus D_4(\sqrt{-3})$	$2^{11}\cdot 3^9\cdot 5\cdot 7$	324
4	$\left\langle A_7\oplus D_5(2)\oplus\langle x\rangle,\ \sqrt{-3}\,y_1+y_2+\frac{1}{8}x,\ (1,0,0,0,0)+\frac{1}{2}x\right\rangle$, $y_1=\frac{1}{8}(-7,1,\ldots,1)$, $y_2=\frac{1}{2}(1,1,1,1,1)$, $(x,x)=8$	$A_7\oplus D_5(2)$	$2^{15}\cdot 3^4\cdot 5^2\cdot 7$	288
5	$\langle D_7(\sqrt{-3})\oplus U_5\oplus\langle x\rangle,\ y_1+y_2+\frac{1}{6}x\rangle$, $y_1=\frac{1}{\sqrt{-3}}(1,\ldots,1)$, $y_2=\frac{1}{2\sqrt{-3}}(-5,1,1,1,1,1)$, $(x,x)=6$	$D_7(\sqrt{-3})\oplus U_5$	$2^{12}\cdot 3^{13}\cdot 5^2\cdot 7$	648
6	$\langle D_7(2)\oplus D_5(\sqrt{-3})\oplus\langle x\rangle,\ (2+\sqrt{-3})y_1+y_2+\frac{1}{12}x\rangle$, $y_1=\frac{1}{2}(1,\ldots,1)$, $y_2=\frac{1}{\sqrt{-3}}(1,1,1,1,1)$, $(x,x)=12$	$D_7(2)\oplus D_5(\sqrt{-3})$	$2^{14}\cdot 3^8\cdot 5^2\cdot 7$	432
7	$\langle A_6\oplus A_6\oplus\langle x\rangle,\ y_1+y_2+\frac{3}{7}x,\ (2+\sqrt{-3})(y_1-y_2)\rangle$, $y_i=\frac{1}{7}(-6,1,1,1,1,1,1)$, $(x,x)=7$	$A_6\oplus A_6$	$2^{10}\cdot 3^5\cdot 5^2\cdot 7^2$	252
8	Complex conjugate of 7			
9	$\langle E_6\oplus E_6\oplus\langle x\rangle,\ y_1+y_2+\frac{1}{3}x,\ \sqrt{-3}\,(y_1-y_2)\rangle$, $y_i\in E_6^*-E_6$, $(x,x)=3$	$E_6\oplus E_6$	$2^{16}\cdot 3^{10}\cdot 5^2$	432
10	$\left\langle A_5\oplus D_4(2)\oplus D_3(\sqrt{-3})\oplus\langle x\rangle,\ \frac{1}{2}(\sqrt{-3},1,1,1)+\frac{1}{\sqrt{-3}}(1,1,1)+\frac{1}{2\sqrt{-3}}x,\ y_1+(1,0,0,0)+(1,0,0)+\frac{1}{6}x\right\rangle$, $y_1=\frac{1}{6}(-5,1,1,1,1,1)$, $(x,x)=6$	$A_5\oplus D_4(2)\oplus D_3(\sqrt{-3})$	$2^{12}\cdot 3^8\cdot 5$	216
11	$\left\langle A_4\oplus A_4\oplus A_4\oplus\langle x\rangle,\ y_1+y_2+\sqrt{-3}\,y_3,\ y_1-y_2+\frac{2\sqrt{-3}}{5}x\right\rangle$, $y_i=\frac{1}{5}(-4,1,1,1,1)$, $(x,x)=5$	$A_4\oplus A_4\oplus A_4$	$2^{11}\cdot 3^4\cdot 5^3$	180
12	$\langle D_4(\sqrt{-3})\oplus D_4(\sqrt{-3})\oplus D_4(\sqrt{-3})\oplus\langle x\rangle,\ y_1+y_2+y_3,\ y_2-y_3+\frac{1}{3}x\rangle$, $y_i=\frac{1}{\sqrt{-3}}(1,1,1,1)$, $(x,x)=3$	$D_4(\sqrt{-3})^3$	$2^{11}\cdot 3^{14}$	324

(*continued on next page*)

Table 1 (*continued*)

No	L	$L^{(2)}$	$\lvert G(L)\rvert$	$\mu_2(L)$
13	$\langle A_3 \oplus A_3 \oplus A_3 \oplus A_3 \oplus \langle x\rangle,$ $y_1+y_2+y_3+y_4,\ 2(y_1+\omega^2 y_2+\omega y_3),$ $y_1+\omega y_2+\omega^2 y_3+\frac{\sqrt{-3}}{4}x\rangle,$ $y_i=\frac{1}{4}(-3,1,1,1)$, $(x,x)=4$	${A_3}^4$	$2^{15}\cdot 3^6$	144
14	$\langle A_2\oplus A_2\oplus A_2\oplus A_2\oplus A_2\oplus A_2\oplus\langle x\rangle,$ $y_1+y_3+y_4+\sqrt{-3}\,y_5+\sqrt{-3}\,y_6,$ $y_2+\sqrt{-3}\,y_3+\sqrt{-3}\,y_4+y_5+y_6,$ $y_3-y_4+y_5-y_6+\frac{1}{3}x,$ $\sqrt{-3}(y_3-y_4-y_5+y_6)\rangle,$ $y_i=\frac{1}{3}(-2,1,1)$, $(x,x)=3$	${A_2}^6$	$2^9\cdot 3^{10}$	108

2. Statement of results

Let ω be a primitive cube root of 1. Then $\mathbb{Z}[\omega]$ is the ring of integers in the field $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Z}[\omega]$ is a principal ideal domain. Thus every finitely generated torsion-free $\mathbb{Z}[\omega]$-module is free. Let V be a vector space over $\mathbb{Q}(\sqrt{-3})$ with a positive definite hermitian product $(\,,)$. A lattice L in V is a finitely generated $\mathbb{Z}[\omega]$-module contained in V such that L contains a basis of V and $(x,y)\in\mathbb{Z}[\omega]$ for all $x,y\in L$.

The discriminant $d(L)$ of L is the determinant of an hermitian matrix corresponding to the form on L. L is unimodular if $d(L)=1$.

Let $L^{(2)}$ denote the sublattice of L generated by all x in L with $(x,x)=2$. Let $\mu_2(L)$ denote the number of x in L with $(x,x)=2$.

Let $G(L)$ denote the group of all automorphisms of L which preserve the form. Then $G(L)$ is finite.

Let I_n denote a lattice of rank n with an orthonormal basis. Equivalently

$$I_n=\bigl\langle (a_1,\dots,a_n)\mid a_i\in\mathbb{Z}[\omega]\bigr\rangle$$

and if $x=(a_1,\dots,a_n)$, $y=(b_1,\dots,b_n)$ then $(x,y)=\sum_{i=1}^n a_i\bar{b}_i$.

We recall some special lattices. $A_n\subseteq I_{n+1}$ is defined by

$$A_n=\left\langle (a_1,\dots,a_{n+1})\;\middle|\; a_i\in\mathbb{Z}[\omega],\ \sum_{i=1}^{n+1}a_i=0\right\rangle.$$

For $\alpha\in\mathbb{Z}[\omega]$ define $D_n(\alpha)\subseteq I_n$ by

$$D_n(\alpha)=\left\langle (a_1,\dots,a_n)\;\middle|\; a_i\in\mathbb{Z}[\omega],\ \sum_{i=1}^{n}a_i\equiv 0 \pmod{\alpha}\right\rangle,$$

$$U_5=\bigl\langle A_5,\bigl(1/\sqrt{-3}\bigr)\bigl(1,\omega,\omega^2,1,\omega,\omega^2\bigr)\bigr\rangle.$$

Finally, E_6 is the lattice generated by corresponding root system.

In order to define the unimodular lattice L_0 without roots consider 14-dimensional space

$$W = \langle (c_1, \dots, c_{14}) \mid c_i \in \mathbb{F}_4 \rangle$$

over a field $\mathbb{F}_4$ of four elements. The group $\mathrm{PSL}_2(13)$ acts on W by permutations of basic elements. The space W has two natural submodules:

$$W^0 = \left\langle (c_1, \dots, c_{14}) \,\middle|\, \sum_{i=1}^{14} c_i = 0 \right\rangle, \qquad W' = \langle (1, \dots, 1) \rangle.$$

All the $\mathrm{PSL}_2(13)$-invariant nontrivial subspaces in W are W', W^0 and two more subspaces, C_1 and C_2, where $\mathrm{PGL}_2(13)$ permutes them [5,6,11]. The subspaces C_1 and C_2 are called extended quadratic residue codes of length 14. The minimum distance of C_j is 6.

Let φ denote the composition mapping $\mathbb{Z}[\omega] \to \mathbb{Z}[\omega]/2\mathbb{Z}[\omega] \cong \mathbb{F}_4$. Define lattices

$$\Gamma_j = \langle (a_1, \dots, a_{14}) \in A_{13} \mid (\varphi(a_1), \dots, \varphi(a_{14})) \in C_j \rangle,$$

$$\Lambda_j = (2 + \sqrt{-3}) \cdot \Gamma_j + \langle (-13, 1, \dots, 1) \rangle.$$

Lattices Λ_1 and Λ_2 are isometric. We denote by L_0 one of the lattices $(1/\sqrt{14})\Lambda_j$. The lattice L_0 is 13-dimensional unimodular lattice with minimum norm 3 [1]. Therefore, it is a hermitian extremal lattice and it produces an extremal euclidean 3-modular 26-dimensional lattice (see [3,8,9] for more informations on extremal and modular lattices). This 3-modular lattice is appeared also in [7] as its automorphism group is a rational irreducible maximal finite subgroup of $GL_{26}(\mathbb{Q})$.

Theorem 1. *There are* 14 *hermitian indecomposable unimodular lattices over* $\mathbb{Z}[\omega]$ *of dimension* 13. *They are listed in Table* 1. *One of them has no roots, and remaining lattices have root systems of rank* 12.

Proof. Since we have a list of unimodular lattices we need only check the "mass formula" from [4]. It is straightforward to compute $\mu_2(L)$ and $|G(L)|$ in each case. On the other hand, we can use Schiemann's computer program [10] to check our calculations. We should also mention that this program helped us to construct lattices with curious root systems. Let $\mathfrak{G}_n$ be a complete set of representatives of the isomorphisms classes of lattices of rank n and discriminant 1. Let ${\mathfrak{G}_n}^0$ be the set of all those lattices in $\mathfrak{G}_n$ which contain no element x with $(x, x) = 1$. Define

$$M_n = \sum_{L \in \mathfrak{G}_n} \frac{1}{|G(L)|}, \quad M_n(2) = \sum_{L \in \mathfrak{G}_n} \frac{\mu_2(L)}{|G(L)|},$$

$$Y_n = \sum_{L \in {\mathfrak{G}_n}^0} \frac{1}{|G(L)|}, \quad Y_n(2) = \sum_{L \in {\mathfrak{G}_n}^0} \frac{\mu_2(L)}{|G(L)|}.$$

Using formulas from [4] one can calculate

$$M_{13} = \frac{630862444823}{2^{23} \cdot 3^{18} \cdot 5^3 \cdot 11}, \qquad Y_{13} = \frac{38507839151}{2^{16} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13},$$

$$M_{13}(2) = \frac{93958236463}{2^{20} \cdot 3^{15} \cdot 5^2 \cdot 11}, \qquad Y_{13}(2) = \frac{1494233}{2^{12} \cdot 3^{10} \cdot 5^2 \cdot 11}.$$

Now the straightforward check of the mass formula for Y_{13} confirms the completeness of our list. And the mass formula for $Y_{13}(2)$ gives an additional check. □

Note that $M_{13} \approx 0.00000014$, $M_{14} \approx 0.000012$, $M_{15} \approx 0.0045$, $M_{16} \approx 6.57$, $M_{17} \approx 42188.20$. It shows that the unimodular lattices in dimensions 14 and 15 could be also classified in the near future.

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